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Lower order terms for the one-level density of elliptic curve L -functions[☆]

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ABSTRACT

It is believed that, in the limit as the conductor tends to infinity, correlations between the zeros of elliptic curve L -functions averaged within families follow the distribution laws of the eigenvalues of random matrices drawn from the orthogonal group. For test functions with restricted support, this is known to be the true for the one- and two-level densities of zeros within the families studied to date. However, for finite conductor Miller's experimental data reveal an interesting discrepancy from these limiting results. Here we use the L -functions ratios conjectures to calculate the 1-level density for the family of even quadratic twists of an elliptic curve L -function for large but finite conductor. This gives a formula for the leading and lower order terms up to an error term that is conjectured to be significantly smaller. The lower order terms explain many of the features of the zero statistics for relatively small conductor and model the very slow convergence to the infinite conductor limit. However, our main observation is that they do not capture the behaviour of zeros in the important region very close to the critical point and so do not explain Miller's discrepancy. This therefore implies that a more accurate model for statistics near to this point needs to be developed.

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1. Introduction

The conjecture that the limiting statistical properties of the zeros of L -functions may be modeled by those of the eigenvalues of random matrices goes back to Montgomery [Mon73], who introduced it in the context of the Riemann zeta-function. For the Riemann zeros this conjecture is supported by extensive numerical [Od197] and theoretical [Mon73, Hej94, BK95, BK96b, RS96] calculations. The generalization to zero statistics within families of L -functions was developed by Katz and Sarnak [KS99a, KS99b], and again there is much evidence supporting it [Rub01]. Random matrix models for the moments of the Riemann zeta-function on its critical line and for central values of L -functions within families were introduced by Keating and Snaith [KS00b, KS00a], and have since been developed extensively [CF00, CFK+05, GHK07, BK07, BK08, CFK+08]. For more background, see [Me05].

The random-matrix moment conjectures extend naturally to ratios of L -functions. The L -functions ratios conjectures were stimulated by the work of Farmer, who, in 1995, made a conjecture for shifted moments of the Riemann zeta-function [Far95]. Nonnenmacher and Zirnbauer [NZ] found formulas for the ratios of characteristic polynomials of random matrices coming from one of the classical compact groups. This was formalised and written up by Conrey, Farmer and Zirnbauer [CFZb] and lead to the development of corresponding ratios conjectures for L -functions in number theory [CFZa].

The Birch and Swinnerton-Dyer conjecture asserts that the rank of an elliptic curve is equal to the order of vanishing at the central point of the associated L -function. The idea of using random matrix theory to predict the frequency of non-zero rank in families of elliptic curves was introduced by Conrey, Keating, Rubinstein and Snaith [CKRS02, CKRS06]. An interesting extension of this is to find a random matrix model for elliptic curve L -functions of a given order of vanishing at the critical point. The first steps in this direction have been taken by Snaith [Sna05] and Miller/Dueñez [Mil06], but it is clear from Miller's numerical computations that there is a still simpler problem concerning the zero statistics of families of rank zero curves that is far from being understood. This problem is the main motivation for the work we shall report on here.

According to the Katz and Sarnak philosophy [KS99a, KS99b], zeros of families of L -functions show the same statistical behaviour as eigenvalues of random matrices drawn from one of the classical compact groups. The zeros of a family of elliptic curve L -functions with even (odd) functional equation should follow the distribution laws of eigenvalues of the even (odd) orthogonal group. Rigorous calculations [Mil02, Mil04, You06] show that as the conductor (the parameter that orders L -functions within a family) tends to infinity, the one- and two-level densities do indeed tend to the expected orthogonal forms for several different families of elliptic curves. That is, as the conductor tends to infinity, the zero statistics approach the scaling limit for large matrix size of the corresponding statistic for the eigenvalues of matrices from $SO(2N)$ or $SO(2N+1)$. (Similar agreement with random matrix theory is shown for many other families of L -functions, see for example [DM06, FI03, Gü105, HR03, HM07, ILS00, ÖS99, RRB, Roy01, Rub01].) The test functions involved in these calculations have a limited range of support, but nonetheless the evidence is compelling. Thus it was surprising to see in Miller's numerical results [Mil06] a distinct repulsion of the zeros from the central point for a family of L -functions of rank 0 elliptic curves, because no repulsion is seen in the statistics of $SO(2N)$ eigenvalues. Of course, in numerical computations the conductor is finite, and so it is clear that an explanation is needed for finite conductor statistics and how they approach the limiting $SO(2N)$ statistic.

We do have a relatively complete understanding of the way in which the random matrix limit is approached for the zero statistics of the Riemann zeta function at a height T up the critical line as $T \rightarrow \infty$. Berry first wrote down an approximate formula describing the finite- T corrections to the random matrix limiting form for a statistic related to the 2-point correlation function in [Ber88] and showed that this described Odlyzko's data remarkably accurately. Later, a formula that is believed to capture all of the essential features was derived by Bogomolny and Keating [BK96a]. The terms in the Bogomolny–Keating formula that describe the corrections to the random matrix limit are often referred to as *lower order terms*. See [BK99] for an overview and numerical illustrations. More recently, Conrey and Snaith [CS07, CS] have shown how the Bogomolny–Keating formula and its extension to all n -point correlation functions can be recovered from the L -functions ratios conjectures [CFZa]. There have also been investigations of lower order terms in the zero statistics of various families of L -functions [FI03, Mil08, Mil, RRB, You05]. These indicate that convergence is on the scale of the logarithm

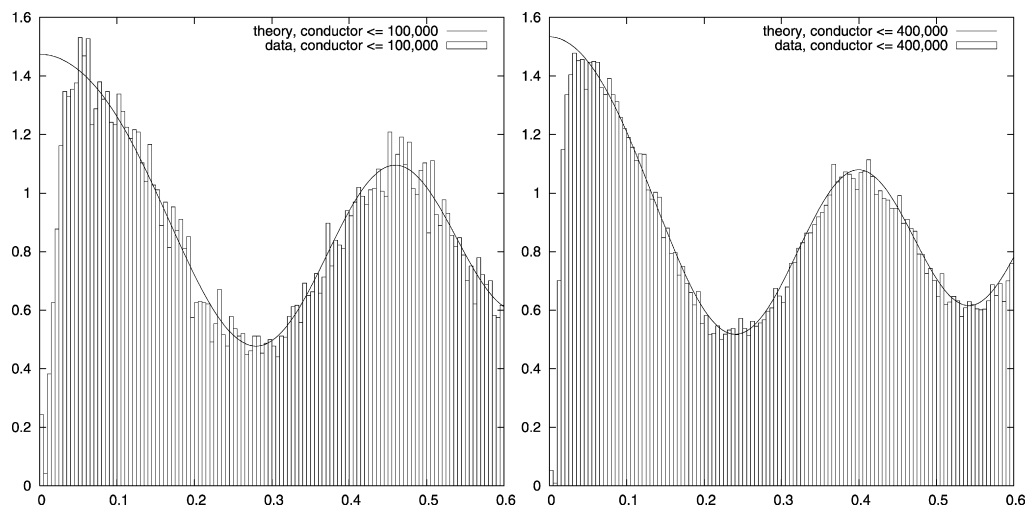


Fig. 1. 1-level density of unscaled zeros from 0 up to height 0.6 of even quadratic twists of $L_{E_{11}}$ with $0 < d \leq 100,000$ for left and $0 < d \leq 400,000$ for right-hand side, prediction (solid), from (2.52), versus numerical data (bar chart).

of the conductor, which makes numerical confirmation difficult unless all the lower order terms are known. Conrey and Snaith have shown how such terms can also be recovered from the ratios conjectures [CS07]. It is thus natural in this context to seek the explanation for the surprising discrepancy observed by Miller in these lower order terms.

In this paper we examine lower order terms in the 1-level density of the zeros of a family of elliptic curve L -functions. Specifically, we investigate even quadratic twists of an elliptic curve L -function, for which we calculate the zeros numerically with Rubinstein's `lcalc` [Rub]. Using the ratios conjectures, in Section 2 we derive a formula for the 1-level density that describes convincingly the intricate structure of the numerical data away from the central point and so explains the rate of approach to the random matrix limit in this region. This comparison with numerical data is carried out in Section 3. However, most interestingly, our formula fails to describe the region very close to the central point. To illustrate our main results, we plot in Fig. 1 a numerical evaluation of the 1-level density together with our formula. Miller's discrepancy corresponds to the region near the origin. Our main conclusion here is then that the explanation for the zero distribution in this region lies beyond the models combining random matrix theory and arithmetical lower order terms considered so far; that is, these formulae are not sufficient to explain the discrepancy. At the end of Section 3 we show that there is preliminary evidence that the discrepancy is contained in the error term of the 1-level density formula. Thus the present work does not imply that formula is wrong, just that important information concerning uniformity near to the origin may still be out of reach in the error term. In fact, it is expected [HM] that the leading and lower order terms of our formula will be reproduced by a rigorous analysis along the lines of [Mil08, Mil, Sto]. However, we note that while such calculations do not rely on any conjecture they, like all the other rigorous results so far, are restricted to a very significantly smaller class of test functions than are allowed for by the ratios conjecture method. In addition, we plan to explore augmented random matrix models that build on the present calculation to explain the phenomenon observed at the origin of Fig. 1 in a future paper with E. Dueñez and S. J. Miller.

2. The 1-level density formula

Let the L -function $L_E(s)$ associated with an elliptic curve E be given by the Dirichlet series

$$L_E(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad (2.1)$$

where the coefficients $(\lambda(n) = a(n)/\sqrt{n})$, with $a_p = p + 1 - \#E(\mathbb{F}_p)$, $\#E(\mathbb{F}_p)$ being the number of points on E counted over \mathbb{F}_p) have been normalised so that the functional equation relates s to $1 - s$:

$$L_E(s) = \omega(E) \left(\frac{2\pi}{\sqrt{M}} \right)^{2s-1} \frac{\Gamma(3/2 - s)}{\Gamma(s + 1/2)} L_E(1 - s). \quad (2.2)$$

Here M is the conductor of the elliptic curve E ; we will consider only prime M . Also, $\omega(E)$ is $+1$ or -1 resulting, respectively, in an even or odd functional equation for L_E .

Let $L_E(s, \chi_d)$ denote the L -function obtained by twisting $L_E(s)$ quadratically. Here d is a fundamental discriminant, i.e., $d \in \mathbb{Z} - \{1\}$, s.t. $p^2 \nmid d$ for all odd primes p and $d \equiv 1 \pmod{4}$ or $d \equiv 8, 12 \pmod{16}$, and χ_d is the Kronecker symbol. Then the twisted L -function (which is itself the L -function associated with another elliptic curve E_d) is given by

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\lambda(n) \chi_d(n)}{n^s} = \prod_p \left(1 - \frac{\lambda(p) \chi_d(p)}{p^s} + \frac{\psi_M(p) \chi_d(p)^2}{p^{2s}} \right)^{-1} \quad (2.3)$$

where ψ_M is the principal Dirichlet character of modulus M :

$$\psi_M(p) = \begin{cases} 1, & \text{if } p \nmid M, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

The functional equation of this L -function is, for $(d, M) = 1$,

$$L_E(s, \chi_d) = \chi_d(-M) \omega(E) \left(\frac{2\pi}{\sqrt{M|d|}} \right)^{2s-1} \frac{\Gamma(3/2 - s)}{\Gamma(s + 1/2)} L_E(1 - s, \chi_d). \quad (2.5)$$

In order to derive the 1-level density of the zeros near the critical point $s = 1/2$ of L -functions in this family of quadratic twists, we consider the average over the family of a ratio of L -functions evaluated at different points:

$$R_E(\alpha, \gamma) := \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{L_E(1/2 + \alpha, \chi_d)}{L_E(1/2 + \gamma, \chi_d)}. \quad (2.6)$$

This is an average over those twisted L -functions that have even functional equations and $0 < d \leq X$. Requiring an even functional equation imposes a restriction on $d \pmod{M}$. We follow the recipe of [CFK+05, CFZa] and the calculations in [CS07] to derive a formula for $R_E(\alpha, \gamma)$ via the ratios conjecture. Note that arriving at a ratios conjecture entails applying a list of manipulations, several of which introduce errors large enough to be significant. The miracle is that these errors appear to cancel out and the recipe yields formulae that have been checked numerically and against specific known cases in many different situations (see [CFZa, CS07]). Recent work of Steven J. Miller [Mil08] has shown that a rigorous calculation of the 1-level density for the family of real quadratic Dirichlet L -functions matches exactly, for a suitably chosen test function, the prediction obtained by applying the ratios recipe. See also [Sto] for further investigations of the ratios conjecture and the 1-level density of the same family of Dirichlet L -functions and [Mil] for Miller's extension of [Mil08] to families of cuspidal new forms.

We use (2.3) to replace $L_E(s, \chi_d)$ in the denominator of (2.6) by

$$\frac{1}{L_E(s, \chi_d)} = \sum_{n=1}^{\infty} \frac{\mu_E(n) \chi_d(n)}{n^s} \quad (2.7)$$

where $\mu_E(n)$ is a multiplicative function defined as

$$\mu_E(n) = \begin{cases} -\lambda(p), & \text{if } n = p, \\ \psi_M(p), & \text{if } n = p^2, \\ 0, & \text{if } n = p^k, k > 2. \end{cases} \quad (2.8)$$

We use the approximate functional equation for the L -function in the numerator of (2.6):

$$L_E(1/2 + \alpha, \chi_d) = \sum_{m < x} \frac{\chi_d(m)\lambda(m)}{m^{1/2+\alpha}} + \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \sum_{n < y} \frac{\chi_d(n)\lambda(n)}{n^{1/2-\alpha}} + \text{remainder}, \quad (2.9)$$

where M is the conductor of the elliptic curve E and $xy = d^2/(2\pi)$. Therefore using the first sum of the approximate functional equation (2.9) we get

$$R_E^1(\alpha, \gamma) := \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \sum_{h, m} \frac{\lambda(m)\mu_E(h)\chi_d(mh)}{m^{1/2+\alpha}h^{1/2+\gamma}}. \quad (2.10)$$

We denote by $R_E^2(\alpha, \gamma)$ the expression that results from using the second sum in the approximate functional equation (2.9). Thus

$$R_E(\alpha, \gamma) \approx R_E^1(\alpha, \gamma) + R_E^2(\alpha, \gamma), \quad (2.11)$$

where the symbol \approx is to remind the reader that we are performing a series of manipulations (called in [CFK+05] the *recipe*) that at each individual step involve large errors, which are believed ultimately to cancel each other. The ratios recipe now calls for a replacement of $\chi_d(mh)$ with its average over the family (the set of d 's being summed over). We set

$$X^* = \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} 1 \quad \text{and} \quad X_b^* = \sum_{\substack{0 < d \leq X \\ d \equiv b \pmod{M}}} 1 \quad (2.12)$$

as the number of fundamental discriminants below X that we are summing over and note (see [CFK+05], Theorem 3.1.1)

$$\frac{1}{X_b^*} \sum_{\substack{0 < d \leq X \\ d \equiv b \pmod{M}}} \chi_d(n) \approx \begin{cases} \chi_b(g)a(n), & \text{if } n = g\Box, \text{ with } (\Box, M) = 1 \text{ and if all} \\ & \text{prime factors of } g \text{ are factors of } M, \\ 0, & \text{otherwise,} \end{cases} \quad (2.13)$$

where

$$a(n) = \prod_{p|\Box} \frac{p}{p+1}. \quad (2.14)$$

This is to say that terms not of the form $n = g\Box$ can be disregarded (this is the so-called ‘harmonic detector’ which is mentioned in [CFK+05]). Since we are considering only curves with prime conductor M , g is simply a power of M . Note that in the cases we are interested in $\chi_b(g) = \omega_E^\ell$ for $g = M^\ell$ because d has been chosen such that $\chi_b(M) = \chi_d(M) = \omega_E$ (we have $\chi_d(M) = \chi_d(-M)$ since we are considering only positive d).

Concentrating on R_E^1 , we replace $\chi_d(mh)$ with the average given by (2.13) and so restrict the sum as follows:

$$R_E^1(\alpha, \gamma) \approx X^* \sum_{hm=\square M^\ell} \frac{\lambda(m)\mu_E(h)a(mh)\omega_E^\ell}{m^{1/2+\alpha}h^{1/2+\gamma}}, \quad (2.15)$$

with $(\square, M) = 1$ and g divisible only by primes dividing M . We write this sum as an Euler product (for convenience denoting by h the exponent on primes dividing h in the sum above and similarly for m) and note that if $m+h \geq 1$ then $a(p^{m+h}) = p/(p+1)$ for primes not dividing the conductor, whereas $a(p^{m+h}) = 1$ if the prime does divide the conductor. So we obtain

$$R_E^1(\alpha, \gamma) \approx X^* V_{\mid}(\alpha, \gamma) V_{\nmid}(\alpha, \gamma) \quad (2.16)$$

where

$$V_{\nmid}(\alpha, \gamma) := \prod_{p \nmid M} \left(1 + \frac{p}{p+1} \sum_{\substack{m, h \geq 0 \\ m+h > 0 \\ m+h \text{ even}}} \frac{\lambda(p^m)\mu_E(p^h)}{p^{m(1/2+\alpha)+h(1/2+\gamma)}} \right) \quad (2.17)$$

$$V_{\mid}(\alpha, \gamma) := \prod_{p \mid M} \left(\sum_{h, m \geq 0} \frac{\lambda(p^m)\mu_E(p^h)\omega_E^{m+h}}{p^{m(1/2+\alpha)+h(1/2+\gamma)}} \right). \quad (2.18)$$

Since $\mu_E(p^h) = 0$ for most powers of p , we only need to consider $h = 0, 1, 2$ in the sum in (2.17) and $h = 0, 1$ in (2.18). Then the Euler products become

$$V_{\nmid}(\alpha, \gamma) = \prod_{p \nmid M} \left(1 + \frac{p}{p+1} \left(\sum_{m=1}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} \right) \right) \quad (2.19)$$

and

$$V_{\mid}(\alpha, \gamma) = \prod_{p \mid M} \left(\sum_{m=0}^{\infty} \left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)+1/2+\gamma}} \right) \right). \quad (2.20)$$

We now factor out the divergent part of R_E^1 using the Riemann zeta function and also, for convenience, we will factor out the symmetric square L -function associated with L_E . This leaves us with a convergent Euler product. In the following, for simplicity, we shall only deal with elliptic curves with prime conductor, M . Recall that the Euler product of a Hasse–Weil L -function $L_E(s)$ coming from the elliptic curve E , with Dirichlet coefficients $\lambda(n)$ normalised so that the functional equation relates s to $1-s$, has the form

$$L_E(s) = \prod_{p \mid M} (1 - \lambda(p)p^{-s})^{-1} \prod_{p \nmid M} (1 - \lambda(p)p^{-s} + p^{-2s})^{-1}. \quad (2.21)$$

Now we can write this product as

$$L_E(s) = \prod_p (1 - \alpha(p)p^{-s})^{-1} (1 - \beta(p)p^{-s})^{-1}, \quad (2.22)$$

where

$$\alpha(p) + \beta(p) = \lambda(p) \quad (2.23)$$

and

$$\alpha(p)\beta(p) = \begin{cases} 0 & \text{for } p|M, \\ 1 & \text{for } p \nmid M. \end{cases} \quad (2.24)$$

Let $L_E(\text{sym}^2, s)$ denote the symmetric square L -function. Then by definition (see [Iwa97, p. 251])

$$L_E(\text{sym}^2, s) = \prod_p (1 - \alpha^2(p)p^{-s})^{-1} (1 - \alpha(p)\beta(p)p^{-s})^{-1} (1 - \beta^2(p)p^{-s})^{-1}. \quad (2.25)$$

We have (see [Con05, p. 236])

$$\lambda(m)\lambda(n) = \sum_{\substack{d|(m,n) \\ (d,M)=1}} \lambda(mn/d^2) \quad (2.26)$$

(where M is the conductor of E) and in particular we have for $p \nmid M$,

$$\lambda(p)^2 = \lambda(p^2) + 1, \quad (2.27)$$

$$\lambda(p^{2m+1})\lambda(p) = \lambda(p^{2m+2}) + \lambda(p^{2m}). \quad (2.28)$$

We wish to write the Euler product in (2.25) in terms of $\lambda(p)$, so we start by using (2.23) to obtain

$$L_E(\text{sym}^2, s) = \prod_p \left(1 - \frac{\lambda(p)^2 - \alpha(p)\beta(p)}{p^s} + \frac{\alpha(p)\beta(p)(\lambda(p)^2 - \alpha(p)\beta(p))}{p^{2s}} - \frac{(\alpha(p)\beta(p))^3}{p^{3s}} \right)^{-1}. \quad (2.29)$$

We now distinguish between $p|M$ and $p \nmid M$, and so, using (2.27) and (2.24), we have

$$L_E(\text{sym}^2, s) = \prod_{p|M} \left(1 - \frac{\lambda(p)^2}{p^s} \right)^{-1} \prod_{p \nmid M} \left(1 - \frac{\lambda(p)^2}{p^s} + \frac{\lambda(p)^2}{p^{2s}} - \frac{1}{p^{3s}} \right)^{-1}. \quad (2.30)$$

Now we reconsider the Euler products in (2.19) and (2.20). In constructing ratios conjectures we usually allow $-\frac{1}{4} < \text{Re}\alpha < \frac{1}{4}$ and $\log X \ll \text{Re}\gamma < \frac{1}{4}$, where the bounds at $\pm\frac{1}{4}$ allow us to control the convergence of Euler products of the type (2.19). In fact, in this application the real parts of α and γ can be considered as very small. Thus we can write

$$\begin{aligned} V_{\nmid M}(\alpha, \gamma) &= \prod_{p \nmid M} \left(1 + \frac{p}{p+1} \left(\sum_{m=1}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} \right) \right) \\ &= \prod_{p \nmid M} \left(1 + \frac{\lambda(p^2)}{p^{1+2\alpha}} - \frac{\lambda(p^2)+1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} + \dots \right), \end{aligned} \quad (2.31)$$

where the ... indicate terms that converge like $1/p^2$ when α and γ are small. We now use the following approximations to factor out the divergent or slowly converging terms. By (2.30) we have

$$L_E(\text{sym}^2, 1+2\alpha) = \prod_p \left(1 + \frac{\lambda(p^2)}{p^{1+2\alpha}} + \dots \right) \quad (2.32)$$

and

$$\frac{1}{L_E(\text{sym}^2, 1 + \alpha + \gamma)} \frac{1}{\zeta(1 + \alpha + \gamma)} = \prod_p \left(1 - \frac{\lambda(p^2) + 1}{p^{1+\alpha+\gamma}} + \dots \right). \quad (2.33)$$

Also, since there is only one prime that divides the conductor M , a factor of $\zeta(1 + 2\gamma)$ will account for the divergence of the term $\frac{1}{p^{1+2\gamma}}$ in (2.31).

Hence we can write

$$V_{\nmid}(\alpha, \gamma) V_{|}(\alpha, \gamma) = Y_E(\alpha, \gamma) A_E(\alpha, \gamma), \quad (2.34)$$

where

$$Y_E(\alpha, \gamma) = \frac{\zeta(1 + 2\gamma) L_E(\text{sym}^2, 1 + 2\alpha)}{\zeta(1 + \alpha + \gamma) L_E(\text{sym}^2, 1 + \alpha + \gamma)}. \quad (2.35)$$

$A_E(\alpha, \gamma)$ is given by

$$\begin{aligned} A_E(\alpha, \gamma) &= Y_E^{-1}(\alpha, \gamma) \\ &\times \prod_{p \nmid M} \left(1 + \frac{p}{p+1} \left(\sum_{m=1}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} \right) \right) \\ &\times \prod_{p|M} \left(\sum_{m=0}^{\infty} \left(\frac{\lambda(p^m) \omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}} \frac{\lambda(p^m) \omega_E^{m+1}}{p^{m(1/2+\alpha)}} \right) \right) \end{aligned} \quad (2.36)$$

and is analytic as $\alpha, \gamma \rightarrow 0$. Hence, by recalling (2.10), we find

$$R_E^1(\alpha, \gamma) \approx \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} Y_E(\alpha, \gamma) A_E(\alpha, \gamma). \quad (2.37)$$

We obtain the other sum $R_E^2(\alpha, \gamma)$ in (2.11) by using the second term in the approximate functional equation (2.9) and carrying out exactly the same steps as above:

$$R_E^2(\alpha, \gamma) \approx \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} Y_E(-\alpha, \gamma) A_E(-\alpha, \gamma). \quad (2.38)$$

By applying the ratios conjecture recipe, we therefore have the result:

Conjecture 2.1 (*Ratios Conjecture*). For some reasonable conditions such as $-\frac{1}{4} < \text{Re } \alpha < \frac{1}{4}$, $\frac{1}{\log X} \ll \text{Re } \gamma < \frac{1}{4}$ and $\text{Im } \alpha, \text{Im } \gamma \ll X^{1-\varepsilon}$, we have

$$\begin{aligned} R_E(\alpha, \gamma) &= \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{L_E(1/2 + \alpha, \chi_d)}{L_E(1/2 + \gamma, \chi_d)} \\ &= \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(Y_E A_E(\alpha, \gamma) + \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} Y_E A_E(-\alpha, \gamma) \right) + O(X^{1/2+\varepsilon}), \end{aligned}$$

where Y_E and A_E are defined at (2.35) and (2.36), respectively, M is the (prime) conductor of the L -function $L_E(s)$ and ω_E is the sign from its functional equation.

We note that the error term $O(X^{1/2+\varepsilon})$ is part of the statement of the ratios conjecture; the power on X is not suggested by any of the steps used in arriving at the main expression in Conjecture 2.1. At the end of Section 3 we propose that the limited data we have available supports a power saving on the error term, but not necessarily a power of $1/2$.

To calculate the 1-level density we actually need the average of the logarithmic derivative of L -functions in this family, so we note that

$$\sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{L'_E(1/2 + r, \chi_d)}{L_E(1/2 + r, \chi_d)} = \frac{d}{d\alpha} R_E(\alpha, \gamma) \Big|_{\alpha=\gamma=r}. \quad (2.39)$$

Using (2.28) for primes not dividing M and the multiplicativity of $\lambda(p)$ for $p|M$, we get $A_E(r, r) = 1$ and we have, with

$$A_E^1(r, r) = \frac{d}{d\alpha} A_E(\alpha, \gamma) \Big|_{\alpha=\gamma=r}, \quad (2.40)$$

$$\begin{aligned} \frac{d}{d\alpha} Y_E A_E(\alpha, \gamma) \Big|_{\alpha=\gamma=r} &= -\frac{\zeta'(1+2r)}{\zeta(1+2r)} A_E(r, r) + \frac{L'_E(\text{sym}^2, 1+2r)}{L_E(\text{sym}^2, 1+2r)} A_E(r, r) + A_E^1(r, r) \\ &= -\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \frac{L'_E(\text{sym}^2, 1+2r)}{L_E(\text{sym}^2, 1+2r)} + A_E^1(r, r) \end{aligned} \quad (2.41)$$

and

$$\begin{aligned} \frac{d}{d\alpha} \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \{Y_E(-\alpha, \gamma) A_E(-\alpha, \gamma)\} \Big|_{\alpha=\gamma=r} \\ = -\left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2r} \frac{\Gamma(1-r)}{\Gamma(1+r)} \frac{\zeta(1+2r) L_E(\text{sym}^2, 1-2r)}{L_E(\text{sym}^2, 1)} A_E(-r, r). \end{aligned} \quad (2.42)$$

Therefore we have for the logarithmic derivative the following:

Theorem 2.2. Assuming the Ratios Conjecture 2.1 and $\frac{1}{\log X} \ll \text{Re}(r) < \frac{1}{4}$ and $\text{Im}(r) \ll X^{1-\varepsilon}$, the average of the logarithmic derivative over a family of quadratic twists (with even functional equation) of the L -function of an elliptic curve with prime conductor M is

$$\begin{aligned} \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{L'_E(1/2 + r, \chi_d)}{L_E(1/2 + r, \chi_d)} &= \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(-\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \frac{L'_E(\text{sym}^2, 1+2r)}{L_E(\text{sym}^2, 1+2r)} + A_E^1(r, r) \right. \\ &\quad \left. - \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2r} \frac{\Gamma(1-r)}{\Gamma(1+r)} \frac{\zeta(1+2r) L_E(\text{sym}^2, 1-2r)}{L_E(\text{sym}^2, 1)} A_E(-r, r) \right) \\ &\quad + O(X^{1/2+\varepsilon}). \end{aligned} \quad (2.43)$$

Here ω_E is the sign from the functional equation of L_E , $L_E(\text{sym}^2, s)$ is the associated symmetric square L -function (defined at (2.25)), and A_E and A_E^1 are arithmetic factors defined at (2.36) and (2.40), respectively.

Note that although we do not know the dependence of the error term on α , we will not significantly increase its size by differentiating because it is analytic, being the difference of $R_E(\alpha, \gamma)$ and the main term in Conjecture 2.1, both of which are analytic.

Let γ_d denote the ordinate of a generic non-trivial zero of $L_E(s, \chi_d)$. We consider the 1-level density

$$S_1(f) := \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \sum_{\gamma_d} f(\gamma_d) \quad (2.44)$$

where f is some nice test function, say an even Schwartz function. By the argument principle we have

$$S_1(f) = \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'_E(s, \chi_d)}{L_E(s, \chi_d)} f(-i(s - 1/2)) ds \quad (2.45)$$

where (c) denotes a vertical line from $c - i\infty$ to $c + i\infty$ and $3/4 > c > 1/2 + 1/\log X$. The integral on the c -line is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t - i(c - 1/2)) \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \frac{L'_E(1/2 + (c - 1/2 + it), \chi_d)}{L_E(1/2 + (c - 1/2 + it), \chi_d)} dt. \quad (2.46)$$

The sum over d can be replaced by Theorem 2.2. The bounds on the size t coming from the ratios conjecture should not limit us here. It is not entirely known in what range of the parameters the ratios conjecture holds, but the test function f can be chosen to decay sufficiently fast that the tails of the integrand, where the ratios conjecture might fail, will not contribute significantly. (See the 1-level density section of [CS07] for more detailed analysis.) Next we move the path of integration to $c = 1/2$ as the integrand is regular at $t = 0$ and get

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(-\frac{\zeta'(1 + 2it)}{\zeta(1 + 2it)} + \frac{L'_E(\text{sym}^2, 1 + 2it)}{L_E(\text{sym}^2, 1 + 2it)} + A_E^1(it, it) \right. \\ & \quad \left. - \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2it} \frac{\Gamma(1 - it)}{\Gamma(1 + it)} \frac{\zeta(1 + 2it)L_E(\text{sym}^2, 1 - 2it)}{L_E(\text{sym}^2, 1)} A_E(-it, it) \right) dt \\ & \quad + O(X^{1/2+\varepsilon}). \end{aligned} \quad (2.47)$$

For the integral on the line with real part $1 - c$, we use the functional equation

$$L_E(s, \chi_d) = \chi_d(-M)\omega_E X(s, \chi_d) L_E(1 - s, \chi_d) \quad (2.48)$$

with

$$X(s, \chi_d) = \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{1-2s} \frac{\Gamma(3/2 - s)}{\Gamma(s + 1/2)} \quad (2.49)$$

to obtain

$$\frac{L'_E(1-s, \chi_d)}{L_E(1-s, \chi_d)} = \frac{X'(s, \chi_d)}{X(s, \chi_d)} - \frac{L'_E(s, \chi_d)}{L_E(s, \chi_d)}. \quad (2.50)$$

The logarithmic derivative of (2.49) evaluated at $s = 1/2 + \alpha$ is

$$\frac{X'(1/2 + \alpha, \chi_d)}{X(1/2 + \alpha, \chi_d)} = -2 \log \left(\frac{\sqrt{M}|d|}{2\pi} \right) - \frac{\Gamma'}{\Gamma}(1 + \alpha) - \frac{\Gamma'}{\Gamma}(1 - \alpha). \quad (2.51)$$

For the integral on the $(1-c)$ line we change variables $s \rightarrow 1-s$ and use (2.50). We thus obtain finally the following:

Theorem 2.3. Assuming the Ratios Conjecture 2.1, the 1-level density for the zeros of the family of even quadratic twists of an elliptic curve L -function $L_E(s)$ with prime conductor M is given by

$$\begin{aligned} S_1(f) &= \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \sum_{\gamma_d} f(\gamma_d) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(2 \log \left(\frac{\sqrt{M}|d|}{2\pi} \right) + \frac{\Gamma'}{\Gamma}(1+it) + \frac{\Gamma'}{\Gamma}(1-it) \right. \\ &\quad + 2 \left[-\frac{\zeta'(1+2it)}{\zeta(1+2it)} + \frac{L'_E(\text{sym}^2, 1+2it)}{L_E(\text{sym}^2, 1+2it)} + A_E^1(it, it) \right. \\ &\quad \left. \left. - \left(\frac{\sqrt{M}|d|}{2\pi} \right)^{-2it} \frac{\Gamma(1-it)}{\Gamma(1+it)} \frac{\zeta(1+2it)L_E(\text{sym}^2, 1-2it)}{L_E(\text{sym}^2, 1)} A_E(-it, it) \right] \right) dt \\ &\quad + O(X^{1/2+\varepsilon}), \end{aligned} \quad (2.52)$$

where γ_d is a generic zero of $L_E(s, \chi_d)$, f is an even test function as described above, ω_E is the sign from the functional equation of L_E , $L_E(\text{sym}^2, s)$ is the associated symmetric square L -function (defined at (2.25)), and A_E and A_E^1 are arithmetic factors defined at (2.36) and (2.40), respectively.

3. Numerical test

We test our prediction—namely formula (2.52)—for the 1-level density with a concrete example (see Fig. 2). We pick the elliptic curve E_{11} with $(a_1, a_2, a_3, a_4, a_6) = (0, -1, 1, 0, 0)$ in the Weierstraß form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (3.1)$$

giving

$$E_{11}: y^2 + y = x^3 - x^2 \quad (3.2)$$

and consider the even quadratic twists of its associated L -function with fundamental discriminants between 0 and 40,000. We are interested in the 1-level density of unscaled zeros from 0 up to height 30. The numerical data is obtained from Rubinstein's lcalc [Rub]. In the range considered we find 11,135 quadratic twists, of which 5562 are even ones with a total of about 590,170 zeros. In Fig. 2

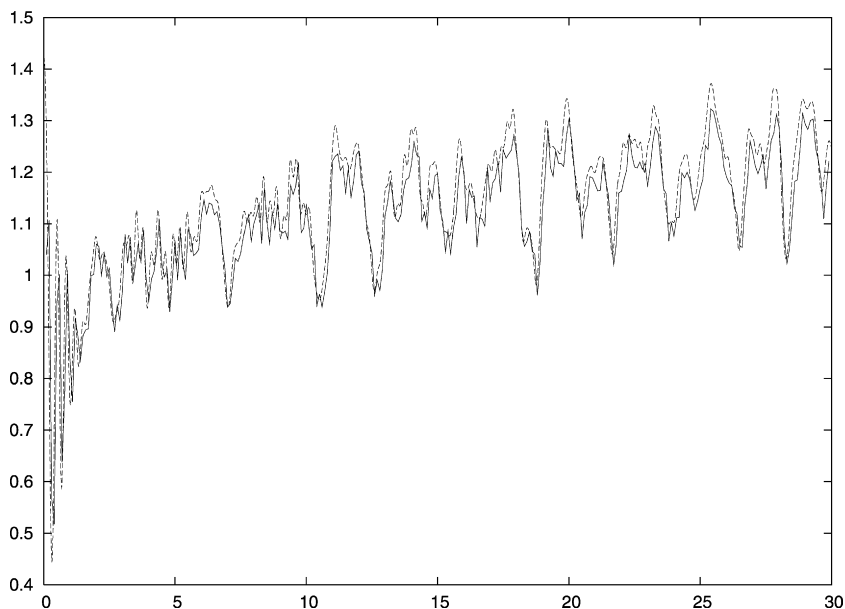


Fig. 2. 1-level density of unscaled zeros from 0 up to height 30 of even quadratic twists of $L_{E_{11}}$ with $0 < d < 40,000$: prediction (dashed), from (2.52), versus numerical data (solid).

we obtain the solid curve from the histogram of this zero data by choosing a binsize of 0.1 and dividing by both the number of quadratic twists with even functional equation, and the mean density of zeros $\log(\sqrt{11}X/(2\pi))$. 593 of the L -functions with even functional equation have (at least) a double zero at the central point; these zeros at the central point are not plotted in Fig. 2. The dashed curve is obtained from the formula (2.52) with $X = 40,000$ and $f(t) = \delta(t - x) + \delta(t + x)$ for x between 0 and 30. This curve is, like the data curve, divided through by the number of quadratic twists with even functional equation and the mean density of zeros. It was computed using a combination of Mathematica and C++. The coefficients $\lambda(p)$ appearing in the arithmetic factor $A_E(\alpha, \gamma)$ were computed using PARI. To compute coefficients of prime powers $\lambda(p^m)$ for $p \nmid M$ the following recursion formulas (see [HM07]) were used

$$\begin{aligned}\lambda(p^{2m}) &= \lambda(p)^{2m} - \sum_{r=0}^{m-1} \left(\binom{2m}{m-r} - \binom{2m}{m-r-1} \right) \lambda(p^{2r}) \\ \lambda(p^{2m+1}) &= \lambda(p)^{2m+1} - \sum_{r=0}^{m-1} \left(\binom{2m+1}{m-r} - \binom{2m+1}{m-r-1} \right) \lambda(p^{2r+1}).\end{aligned}\quad (3.3)$$

In general there is good agreement between the data and the theoretical curve, which captures the main features of the data. We would expect better agreement with a larger set of data, since the data seems not yet to have resolved all the peaks further out along the axis.

A closer look reveals that the 1-level density is strongly governed by the non-trivial zeros of $\zeta(s)$ and $L(\text{sym}^2, s)$: we observe that some dips of the data curve are located at $\gamma/2$ where γ is the ordinate of a non-trivial zero of the Riemann zeta function. This is captured in the term

$$-\frac{\zeta'(1+2it)}{\zeta(1+2it)} \quad (3.4)$$

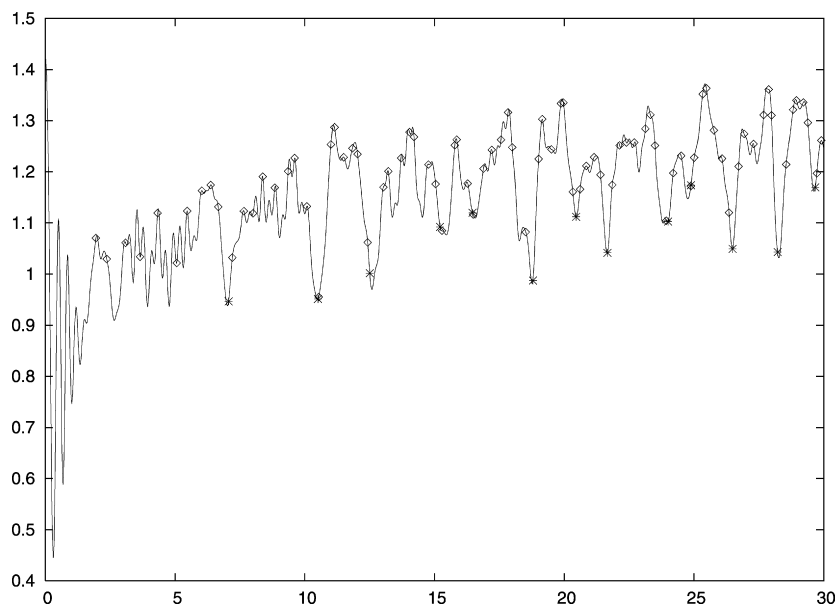


Fig. 3. Effects of non-trivial zeros of the Riemann zeta function (indicated by *) and the non-trivial zeros of $L(\text{sym}^2, s)$ function (indicated by \diamond) on the conjectural formula (2.52) for the 1-level density of unscaled zeros from 0 up to height 30 of even quadratic twists of L_{E11} with $0 < d < 40,000$.

of our conjecture for $S_1(f)$. In Fig. 3 we mark the position of a non-trivial zero of the Riemann zeta function on our conjectural answer by a *. These * are all localised in or around a neighbourhood of a dip. This phenomenon has been encountered before, in the study of lower order terms of the number variance [Ber88] and the correlation functions [BK99, BK96a, CS07, CS08, CS] of the Riemann zeros, and in the one-level density of other families of L -functions [CS07].

On the other hand we observe that some peaks are located at $\tilde{\gamma}/2$ where $\tilde{\gamma}$ is the ordinate of a non-trivial zero of $L_E(\text{sym}^2, s)$. This is captured in the term

$$\frac{L'_E(\text{sym}^2, 1 + 2it)}{L_E(\text{sym}^2, 1 + 2it)} \quad (3.5)$$

of our conjecture for $S_1(f)$. In Fig. 3 we mark the position of a non-trivial zero of $L_E(\text{sym}^2, s)$ by a \diamond . The majority of these \diamond s are localised in or around a neighbourhood of a peak. In particular, we observe that if a zero of the Riemann zeta function is close to a zero of $L(\text{sym}^2, s)$ then these zeros are localised in or around a dip. Hence, zeros of the Riemann zeta function seem to dominate the behaviour of the 1-level density more than the zeros of $L(\text{sym}^2, s)$. This may be explained because the density of the Riemann zeros in this range is smaller than that of the zeros of $L(\text{sym}^2, s)$ and so in terms of the mean zero density the one-line is closer to the half-line in the case of the Riemann zeta function. Therefore one would expect the Riemann zeros to have a larger effect.

The term

$$-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} \frac{\Gamma(1-it)}{\Gamma(1+it)} \frac{\zeta(1+2it)L_E(\text{sym}^2, 1-2it)}{L_E(\text{sym}^2, 1)} A_E(-it, it), \quad (3.6)$$

from (2.52), makes its most obvious contribution by causing the oscillation near the origin of the plot of our conjectural answer for the 1-level density. The factor $(\frac{\sqrt{M}|d|}{2\pi})^{-2it}$ results in oscillations on the scale of the mean density of the zeros of the original L -function, L_E .

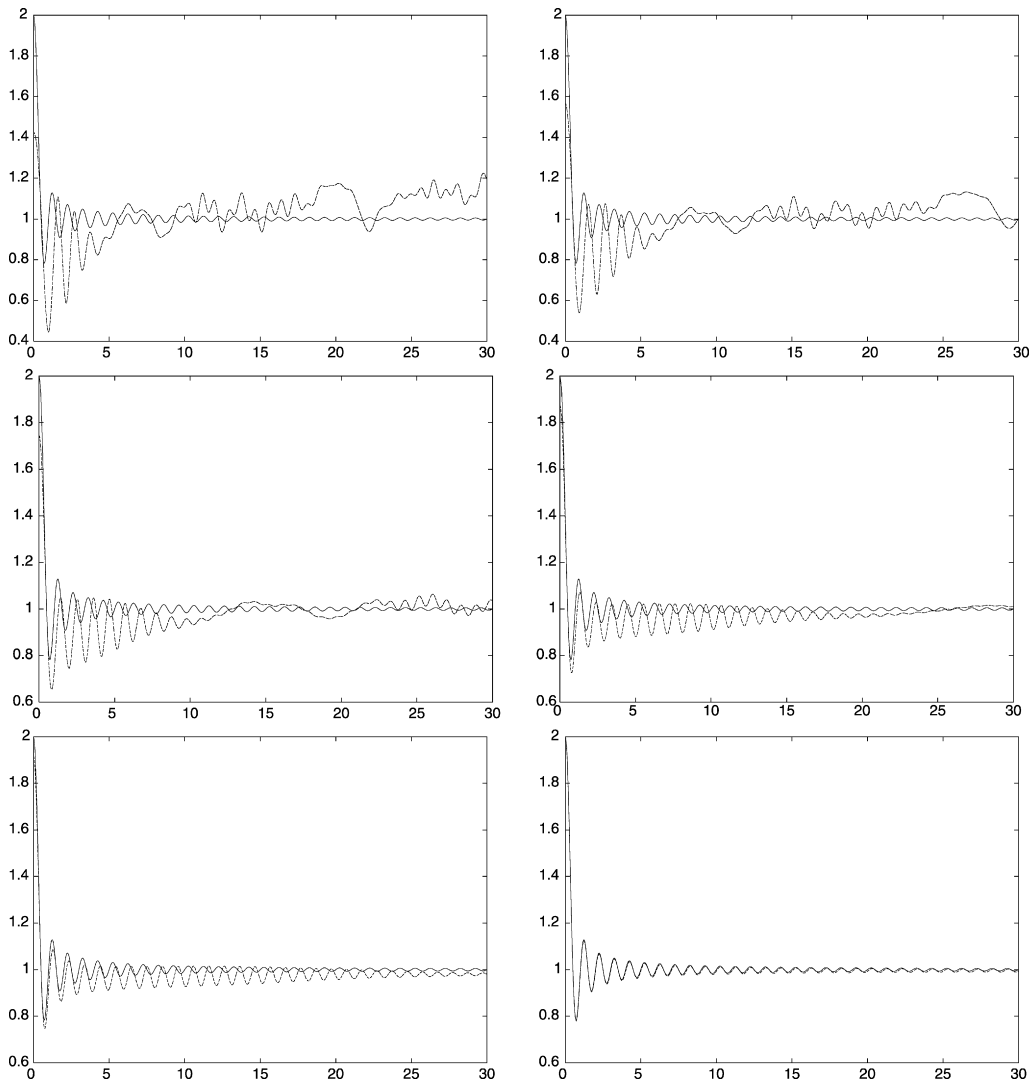


Fig. 4. Scaled limiting 1-level density of $SO(2N)$ (solid) versus scaled formula (3.10) divided by X^* (dashed) for: $0 < d \leq 40,000$ (top left), $0 < d \leq 10^6$ (top right), $0 < d \leq 10^{10}$ (middle left), $0 < d \leq 10^{20}$ (middle right), $0 < d \leq 10^{30}$ (bottom left), $0 < d \leq 10^{300}$ (bottom right).

In summary, we notice that the lower order terms dominate the behaviour of the zeros when we are far from the limit of infinite conductor (in the family of quadratic twists, E_d , the conductor increases like d^2). This becomes more obvious when we compare our conjectural answer for finite conductors with the limiting theoretical result: in Fig. 4 we consider the scaled 1-level density of $SO(2N)$ in the limit $N \rightarrow \infty$ against our conjectural answer (also scaled) for finite conductor. We observe convergence to the limiting theoretical result as we increase X , the cut-off point for d . The observed effects of the arithmetical terms for small and finite conductors are washed out and shifted away from the origin in the large conductor limit.

To further understand the approach to the limiting distribution, we calculate the 1-level density for scaled zeros and recover the limit and the next to leading order term from (2.52). As a first step we rescale the variable t in (2.52) as

$$\tau = t(L/\pi), \quad (3.7)$$

and define

$$f(t) = g(t(L/\pi)), \quad (3.8)$$

where

$$L := \log\left(\frac{\sqrt{M}X}{2\pi}\right), \quad (3.9)$$

and get, after a change of variables,

$$\begin{aligned} \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right) &= \frac{1}{2L} \int_{-\infty}^{\infty} g(\tau) \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(2 \log\left(\frac{\sqrt{M}|d|}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1 + \frac{i\pi\tau}{L}\right)\right. \\ &\quad + \frac{\Gamma'}{\Gamma}\left(1 - \frac{i\pi\tau}{L}\right) + 2\left[-\frac{\zeta'(1 + \frac{2i\pi\tau}{L})}{\zeta(1 + \frac{2i\pi\tau}{L})} + \frac{L'_E(\text{sym}^2, 1 + \frac{2i\pi\tau}{L})}{L_E(\text{sym}^2, 1 + \frac{2i\pi\tau}{L})}\right. \\ &\quad - \left.\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L} \frac{\Gamma(1 - \frac{i\pi\tau}{L})}{\Gamma(1 + \frac{i\pi\tau}{L})} \frac{\zeta(1 + \frac{2i\pi\tau}{L})L_E(\text{sym}^2, 1 - \frac{2i\pi\tau}{L})}{L_E(\text{sym}^2, 1)}\right. \\ &\quad \times \left.A_E\left(-\frac{i\pi\tau}{L}, \frac{i\pi\tau}{L}\right) + A_E^1\left(\frac{i\pi\tau}{L}, \frac{i\pi\tau}{L}\right)\right] d\tau \\ &\quad + O(X^{1/2+\varepsilon}). \end{aligned} \quad (3.10)$$

We write the number of fundamental discriminants less than or equal to X as

$$X^* := \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} 1. \quad (3.11)$$

Using the Euler–Maclaurin formula we make the approximation

$$\sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \log\left(\frac{\sqrt{M}|d|}{2\pi}\right) = X^* \left[\log\left(\frac{\sqrt{M}X}{2\pi}\right) - 1 \right] + O(X^{1/2+\varepsilon}). \quad (3.12)$$

In the same manner we have

$$\sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L} = X^* \left(1 + \frac{2i\pi\tau}{L} + O(L^{-2})\right) e^{-2i\pi\tau} + O(X^{1/2}). \quad (3.13)$$

Writing

$$\zeta(s+1) = \frac{1}{s} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n s^n, \quad (3.14)$$

we have

$$\frac{\zeta'(1+s)}{\zeta(1+s)} = -s^{-1} + \gamma + (-\gamma^2 - 2\gamma_1)s + O(s^2), \quad (3.15)$$

where $\gamma = \gamma_0$ is Euler's constant, and so

$$\zeta\left(1 + \frac{2i\pi\tau}{L}\right) = \frac{L}{2i\pi\tau} + \gamma + O(L^{-1}) \quad (3.16)$$

and

$$\frac{\zeta'(1 + \frac{2i\pi\tau}{L})}{\zeta(1 + \frac{2i\pi\tau}{L})} = -\frac{L}{2i\pi\tau} + \gamma + O(L^{-1}). \quad (3.17)$$

Simple Taylor expansions of the other factors in (3.10) lead us to, with the relation between f and g given in (3.8),

$$\begin{aligned} \frac{1}{X^*} S_1(f) &= \frac{1}{X^*} \sum_{\substack{0 < d \leq X \\ \chi_d(-M)\omega_E = +1}} \sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right) \\ &= \int_{-\infty}^{\infty} g(\tau) \left(1 + \frac{\sin(2\pi\tau)}{2\pi\tau} - a_1 \frac{1 + \cos(2\pi\tau)}{L} - a_2 \frac{\pi\tau \sin(2\pi\tau)}{L^2} + O\left(\frac{1}{L^3}\right)\right) d\tau, \end{aligned} \quad (3.18)$$

where

$$a_1 = 1 + 2\gamma - A_E^1(0, 0) - \frac{L'_E(\text{sym}^2, 1)}{L_E(\text{sym}^2, 1)} \quad (3.19)$$

and

$$\begin{aligned} a_2 &= 2 + 4\gamma + 3\gamma^2 - 2\gamma_1 + B'(0) + 2\gamma B'(0) \\ &\quad - 2 \frac{L'_E(\text{sym}^2, 1)}{L_E(\text{sym}^2, 1)} - \frac{4\gamma L'(1)}{L(1)} - \frac{B'(0)L'_E(\text{sym}^2, 1)}{L_E(\text{sym}^2, 1)} + \frac{B''(0)}{4} + \frac{L''_E(\text{sym}^2, 1)}{L_E(\text{sym}^2, 1)}, \end{aligned} \quad (3.20)$$

with

$$B'(0) = \frac{d}{dr} A_E(-r, r)|_{r=0} \quad \text{and} \quad B''(0) = \frac{d^2}{dr^2} A_E(-r, r)|_{r=0}. \quad (3.21)$$

In order to obtain (3.19) we use the following identity

$$-\frac{1}{2} B'(0) = A_E^1(0, 0). \quad (3.22)$$

We establish identity (3.22) by simple algebra and using (2.28) for primes not dividing M , the multiplicativity of $\lambda(p)$ for $p|M$ and $A_E(r, r) = 1$.

This work was initially conceived to investigate the unexpected numerical results found by Steven J. Miller [Mil06] near the origin of the histogram of the distribution of the first zero above the central point of a family of rank zero L -functions. He observed very few examples of zeros lying close to

the central point. That is, he observed the phenomenon of repulsion of zeros from the central point which we know, from rigorous work on the 1-level and 2-level densities [Mil02,Mil04,You06] does not persist in the large conductor limit. Since the 1-level density (a histogram of all zeros) and the distribution of the lowest zero (a histogram of the lowest zero of each L -function) are the same for very small distances from the central point, it is natural to enquire whether the ratios conjecture yields a formula for the 1-level density which would display and explain Miller's observed repulsion at finite conductor. Although it can be seen from Fig. 4 that the formula (3.10) is significantly smaller near the origin than the limiting curve, and approaches it from below as the conductor increases, there is no evidence of repulsion. This is a major discrepancy from the data, as seen in Fig. 1: away from the critical point we have a nice match between the prediction and the data while near the critical point we find fewer zeros in the data than predicted by our formula. It is most interesting that the main terms of the ratios conjecture do not capture this important feature. Of course, the natural question is whether this contradicts the ratios conjecture, or whether the discrepancy can be accounted for by the error term. As expected due to the limited data available, the test described below is inconclusive, but shows signs that the error term in the ratios conjecture (and hence on the one level density in (2.52)) is of the form $X^{b+\varepsilon}$, for $b < 1$. The ratios conjecture is usually stated with $b = 1/2$.

We fix several sample points at various distances away from the critical point and measure the difference between the main terms of our prediction (that is, the sum over d inside the integral in (2.52)) and the data. In fact, we compare the normalised versions of our prediction and data by dividing through by the number of fundamental discriminants X^* less than X and the mean density of zeros. So let us denote this difference between the main terms of the normalised theory and the data at a fixed height t and fixed X by $\Delta(t, X)$. Since we have divided by X^* , which is proportional to X , this difference is expected to be of size

$$|\Delta(t, X)| = O(X^{b-1+\varepsilon}). \quad (3.23)$$

The quantity we will plot is

$$Q_{\Delta}(t, X) := \frac{\log(|\Delta(t, X)|)}{\log X}, \quad (3.24)$$

and if the ratios conjecture with error term $X^{b+\varepsilon}$ is correct then we would expect

$$Q_{\Delta}(t, X) = b - 1 + O\left(\frac{\log \log X}{\log X}\right) \quad (3.25)$$

as $X \rightarrow \infty$.

In Fig. 5 we plot the quantity $Q_{\Delta}(t, X)$ for $0 < X < 400,000$ and for various fixed sample points $t_1 = 0.01$, $t_2 = 0.02$, $t_3 = 0.03$, $t_4 = 0.04$, $t_5 = 0.05$, $t_6 = 0.4$ and $t_7 = 0.6$. We notice that the curves are much smoother for sample points near the critical point, $t = 0$, e.g. t_1, t_2, t_3 . In the range $0 < X < 400,000$ these points are well inside the region where the zero data shows repulsion at the critical point; see Fig. 1. Thus the difference between the theory (smooth curve in Fig. 1) and data (histogram) does not change sign as X increases. Presumably it is the amplification of such sign changes by the logarithm in (3.24) that is responsible for the jagged curves in Fig. 5 for sample points t_4, t_5 and t_6 .

We see also that the curves at sample points close to the critical point appear at first sight to indicate a larger error term—in fact, over this range of X the t_1 curve implies $b - 1 > 0$! If a limit such as (3.25) exists, it does not seem to behave uniformly in t . However, the t_1, t_2 and t_3 curves are decaying as X increases and we do not have enough data to see what their final behaviour will be. We remember that the convergence is like $\log \log X / \log X$ (which is about 0.2 for $X = 400,000$), so we would need much more data to be able to make a sensible conclusion about the size of the error term.

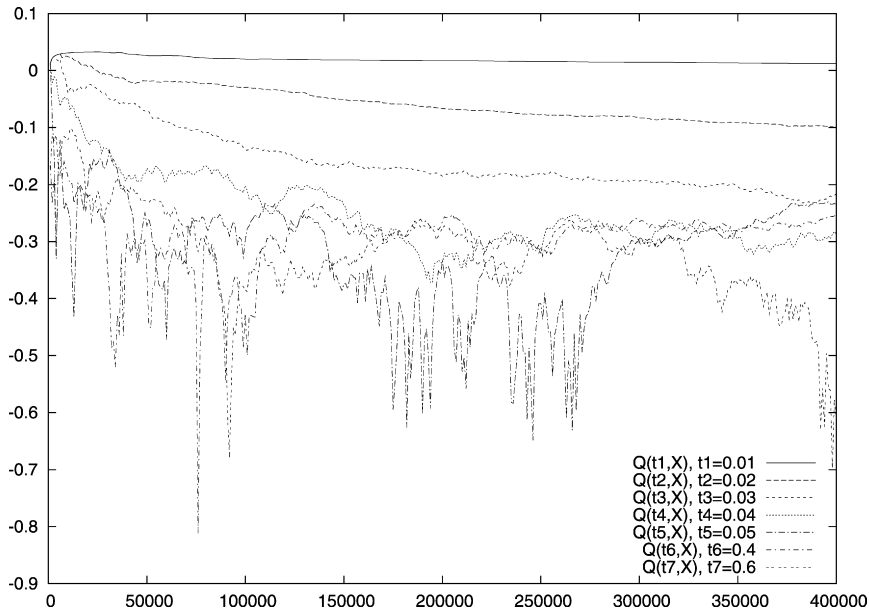


Fig. 5. discrepancy $Q_{\Delta}(t, X)$ from (3.24) between prediction and data for the 1-level density of even quadratic twists of $L_{E_{11}}$ with $0 < d < 400,000$.

Also, it is interesting to note that at the right-hand side of Fig. 5 the $t_3 = 0.03$ curve has decayed to a level comparable to the curves of the sample points that are more distant from $t = 0$. Examining Fig. 1, it appears that the area of major discrepancy between the ratios conjecture prediction and the data (that is, where the data shows repulsion from the critical point at $t = 0$) lies between $t = 0$ and about $t = 0.03$. We expect that this region will narrow as the range of discriminants, d , increases, and this is born out by comparing the two pictures in Fig. 1; the data grows more quickly to the height of the solid curve in the right hand picture where $0 < d < 400,000$, than in the left hand picture where $0 < d < 100,000$. Thus at the right hand edge of Fig. 5, the point $t_3 = 0.03$ is about to move into the region where there is good agreement between the ratios conjecture prediction and the data. Making a speculative conclusion from the limited data available, this suggests that the curves for t_1 and t_2 , or any other fixed t , would also decay to this level if we could gather enough data to shrink the area of discrepancy at the origin of Fig. 1 to a narrow enough band.

It is impossible to say from the available data what the exponent b in the error term of the ratios conjecture is. There is certainly no evidence to suggest $b = 0.5$, but the possibility that the curves in Fig. 5 would decay to -0.5 if we could vastly extend the range of the plot is not ruled out. However, Fig. 5 certainly appears to suggest that $b < 0$ and so the error term is a power of X smaller than the main term.

4. Summary

We find that the ratios conjecture provides a formula for the one level density of zeros of a family of quadratic twists of an elliptic curve L -function that agrees with data for finite conductor, except in the vicinity of the critical point, $t = 0$, and explains the arithmetic nature of the lower order terms which entirely dominate the behaviour of the statistic away from $t = 0$. The ratios conjecture prediction, when properly scaled, approaches the limiting $SO(2N)$ random matrix result as the family of elliptic curves includes those with larger and larger conductor. This supports all the available evidence that $SO(2N)$ is the correct limit for zero statistics in this family. It is very interesting that the ratios conjecture prediction does not capture the phenomenon of zero repulsion from the critical point, $t = 0$, but the data we have available certainly allows for the ratios conjecture to be correct

with some power $b < 1$ of X in the error term; the discrepancy between the ratios conjecture prediction and the data (at the origin of Fig. 1) can quite possibly be contained in the error term. It is also expected [HM] that, along the lines of [Mil08, Mil, Sto], the leading and lower order terms of the ratios prediction can be obtained rigorously for this family of L -functions, with the usual restriction on the test functions used.

In ongoing work of the authors in collaboration with E. Dueñez and S. J. Miller we propose an explanation for the observed repulsion of zeros near the central point for finite conductor and a random matrix model that captures the phenomenon.

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